Math 8 Homework 9

1 Arithmetic and Algebra of Complex Numbers

(a) Simplify the following.

  (a) $(1+i)^{100}$.
  (b) $(1+e^{i\theta})^n$.
  (c) $\exp(e^{i\theta})$

(b) Geometrically describe the set of all points $z \in \mathbb{C}$ such that:

  (i) $|z-1| = 4$
  (ii) $|z-i| = |z-4|$
  (iii) $|z| = \mathrm{Re}(z+2)$

(c) Solve the following equations in $\mathbb{C}$.

  (i) $z^6 - 2z^3 + 2 = 0$
  (ii) $(z+1)^5 = z^5$
  (iii) $e^z = 1+i$
  (iv) $z^4 = 5(z-1)(z^2-z+1)$

(d) Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \frac{4-2\cos\theta}{5-4\cos\theta}$.

(e) Let $p(z)$ be a polynomial with complex coefficients. Prove that $p: \mathbb{C} \to \mathbb{C}$ is surjective.

(f) Define $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix a point $p \in \mathbb{D}$. Prove there exists a function $f : \mathbb{D} \to \mathbb{C}$ of the form

$$f(z) = \frac{az+b}{cz+d}$$

so that $f(p) = 0$ and $|f(z)| \to 1$ as $|z| \to 1$.

2 Problem Solving with Complex Numbers

(a) A regular n-gon is inscribed in a unit circle. Take one vertex and consider the $n-1$ line segments connecting it to the other vertices.

  (a) Prove that the product of their lengths is n.
  (b) What is the sum of the squares of their lengths?

(b) Some positive integers can be written as a sum of two perfect squares (for instance, $25 = 3^2 + 4^2$). Prove that if m and n can each be written as a sum of two squares, so can mn.

(c) Let $\omega \in \mathbb{C}$ be a nonreal cube root of 1. Find a function $f : \mathbb{C} \to \mathbb{C}$ so that

$$f(z) + f(\omega z) = \exp(z)$$

for all $z \in \mathbb{C}$. Prove there is only one such f.

(d) An arithmetic progression is a sequence of the form $a, a+d, a+2d, \ldots$, where d is called the step size.

  (a) Let $a, d \in \mathbb{N}$. Show that $z^a + z^{a+d} + z^{a+2d} + \cdots = \dfrac{z^a}{1-z^d}$ whenever $z \in \mathbb{D}$.
  (b) Suppose we have positive integers a_k, d_k so that each d_k is distinct and

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_n}}{1-z^{d_n}}$$

  for all $z \in \mathbb{D}$. Prove that $n = 1$ and $a_1 = d_1 = 1$.
  (c) Prove that $\mathbb{N}$ cannot be partitioned into a finite collection of arithmetic progressions with distinct step sizes d except in the trivial case when $a = d = 1$.